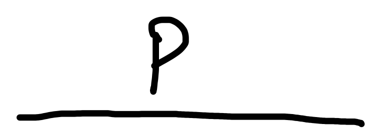
§3. Homotopy Lie algebra and BRST

As we have seen, asymptotic analysis of $\int e^{f/\hbar}$

leads to combinatorial formula via "<u>Graphs</u>"

(Feynman Diagram expansion)

propagator : 

vertex : 

Our next goal is to find its connection w/.

Constructions in homological algebra.

- <u>DGLA</u> (differential graded Lie algebra)

<u>Def'n</u>. A graded Lie algebra is a $\mathbb{Z}$-graded vector space

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$$

w/. a bilinear map $[-,-] : \mathfrak{g}\otimes\mathfrak{g} \longmapsto \mathfrak{g}$

satisfying the following conditions :

a) (graded bracket) $[g_\alpha, g_\beta] \subset g_{\alpha+\beta}$

b) (graded skewsymmetry) $[a,b] = -(-1)^{\alpha\beta}[b,a]$

c) (Jacobi Identity) for $\forall$ $a \in g_\alpha$, $b \in g_\beta$, $c \in g_\gamma$

$$[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]$$

<u>Def'n</u>: A DGLA is a graded Lie algebra g w/. a differential d of deg=1 ($d: g_k \to g_{k+1}$) satisfying

- $d^2 = 0$
- $d[a,b] = [da,b] + (-1)^a [a,db]$ for $a \in g_\alpha$, $b \in g_\beta$.

<u>Eg</u>: An ordinary Lie algebra is a DGLA where

- $g = g_0$ so g is concentraded in deg=0
- $d = 0$

We see DGLA is a natural generalitation of Lie algebras

<u>Eg</u>: Let X be a manifold, $\mathfrak{g}$ a Lie algebra.

Let $(\Omega^{\cdot}(X), d)$ be the de Rham Complex. Then

$$\left(\Omega^{\cdot}(X)\otimes\mathfrak{g},\ d,\ [-,-]_{\mathfrak{g}}\right) \text{ is a DGLA.}$$

- $\Omega^{k}\otimes\mathfrak{g}$: $\deg = k$ component
- $d : \Omega^{k}\otimes\mathfrak{g} \mapsto \Omega^{k+1}\otimes\mathfrak{g}$ de Rham differential

$$d(\alpha\otimes h) = d\alpha\otimes h \quad \text{for } \alpha\in\Omega^{\cdot},\ h\in\mathfrak{g}.$$

- the bracket is induced from the Lie bracket $[-,-]_{\mathfrak{g}}$ on $\mathfrak{g}$

$$[\alpha_1\otimes h_1, \alpha_2\otimes h_2] = (\alpha_1\wedge\alpha_2)\otimes[h_1,h_2]_{\mathfrak{g}}$$

for any $\alpha_1,\alpha_2\in\Omega^{\cdot}$, $h_1,h_2\in\mathfrak{g}$.

This example is related to Chern-Simons theory (CS)

Eg: Let X be a complex manifold. Let

$$(\Omega^{0,\bullet}(X), \bar\partial) \quad \text{Dolbeault complex}$$

Let $T_X^{1,0}$ denote the bundle of $(1,0)$-vector fields.

Then $\left(\Omega^{0,\bullet}(X, T_X^{1,0}), \bar\partial, [-,-]\right)$ is a DGLA.

Explicitly, let $\{z^i\}$ be local holomorphic coordinate

An element $\alpha \in \Omega^{0,k}(X, T_X^{1,0})$ can be written as

deg=k component

$$\alpha = \sum_{i,J} \alpha^i_{\bar J}\, d\bar z^{\bar J} \otimes \partial_{z^i}$$

Here $J = \{j_1 < \cdots < j_k\}$ is a multi-index and

$$d\bar z^{\bar J} = d\bar z^{j_1} \wedge \cdots \wedge d\bar z^{j_k}$$

Then the differential is

$$\bar\partial \alpha = \sum \bar\partial(\alpha^i_{\bar J}) \wedge d\bar z^{\bar J} \otimes \partial_{z^i}$$

$$= \sum \bar{\partial}_{\ell} \alpha^{i}_{\bar{j}} \, d\bar{z}^{\ell} \wedge d\bar{z}^{\bar{j}} \otimes \partial_{z^i}$$

Given two elements

$$\alpha = \sum \alpha^{i}_{\bar{j}} \, d\bar{z}^{\bar{j}} \otimes \partial_i \qquad \beta = \sum \beta^{i}_{\bar{m}} \, d\bar{z}^{\bar{m}} \otimes \partial_i$$

the bracket is given by

$$[\alpha, \beta] = \left(\alpha^{j}_{\bar{j}} \, \partial_j \beta^{i}_{\bar{m}} - \beta^{j}_{\bar{m}} \, \partial_j \alpha^{i}_{\bar{j}} \right) d\bar{z}^{\bar{j}} \wedge d\bar{z}^{\bar{m}} \otimes \partial_i$$

On deg=0 components, this is just the usual Lie bracket on (1,0) vector fields.

As we will study later, this example is related to the deformation of complex structures and also the so-called Kodaira-Spencer gravity

(this is the B-twisted top. closed string field theory)

- Chevalley - Eilenberg and BRST

Let g be a Lie algebra. Let $g^\vee$ be its linear dual.

For simplicity, let us assume g is finite dim'l.

Consider

$$C^\bullet(g) = \bigoplus_k \wedge^k g^\vee$$

This is a polynomial algebra in odd variables.

If we choose basis $\{e_\alpha\}$ of g, and dual basis $\{c^\alpha\}$ of $g^\vee$, then we can write

$$C^\cdot(g) = \mathbb{R}[c^\alpha] \quad \text{where} \quad c^\alpha c^\beta = -c^\beta c^\alpha$$

Let $[-,-]: \wedge^2 g \longmapsto g$ be the Lie bracket.

Taking the dual, we find

$$[-,-]^\vee : g^\vee \longmapsto \wedge^2 g^\vee$$

This defines a derivation on $C^\bullet(\mathfrak{g})$

$$d_{CE} : C^\bullet(\mathfrak{g}) \longmapsto C^\bullet(\mathfrak{g})$$

which is determined by

① on generators : $d_{CE} = [-,-]^\vee$ on $\mathfrak{g}^\vee$

② d_{CE} satisfies the graded Leibnitz rule

$$d_{CE}(a \wedge b) = (d_{CE}\, a) \wedge b + (-1)^k a \wedge d_{CE}(b)$$

$$\text{if } a \in C^k(\mathfrak{g})$$

Prop: $\boxed{d_{CE}^2 = 0}$ So $(C^\bullet(\mathfrak{g}), d_{CE})$ defines a complex, called Chevelley-Eilenberg Complex

In fact, $d_{CE}^2 = 0$ is equivalent to Jacobi-Identity, this is a good exercise.

In terms of the above chosen basis, let

$$[e_\alpha, e_\beta] = \sum_\gamma f_{\alpha\beta}^\gamma e_\gamma$$

Structure constant

Then we have the explicit formula

$$d_{CE}(c^\alpha) = \frac{1}{2}\sum_{\beta,\gamma} f^\alpha_{\beta\gamma} c^\beta c^\gamma$$

This is used in physics to describe the BRST formalism for gauge theory:

$c^\alpha \rightsquigarrow$ ghost

$d_{CE} \rightsquigarrow$ BRST differential

The above construction also generalizes to the case when we have a $\mathfrak{g}$-rep. Such $\mathfrak{g}$-rep is given by matter field in BRST formalism.

- Linear algebra for graded vector spaces

We will generalize the above construction to DGLA, and eventually to Homotopic Lie algebras.

Let us first fix some conventions for graded spaces.

Let $W = \bigoplus_{n\in\mathbb{Z}} W_n$ be a $\mathbb{Z}$-graded vector space.

- $W[n]$ denotes the $\mathbb{Z}$-graded space w/.

$$W[n]_m := W_{n+m} \quad (\text{deg shift by } n)$$

- $W^\vee$ denotes the linear dual w/.

$$W^\vee_m = \mathrm{Hom}(W_{-m}, k)$$ base field

Given two $\mathbb{Z}$-graded vector spaces V, W

$$(V\otimes W)_n = \bigoplus_{i+j=n} (V_i \otimes W_j)$$ (base field is implicit)

$$\mathrm{Hom}(V,W)_n = \bigoplus_i \mathrm{Hom}(V_i, W_{i+n})$$

- $\mathrm{Sym}^m(V)$ = m-th graded symmetric tensor

  $= V^{\otimes m}/\sim$ where $a\otimes b \sim (-1)^{|a||b|}\, b\otimes a$

  $|a|$ is the parity of a.

- $\Lambda^m(V)$ = m-th graded skew-symmetric tensor

  $= V^{\otimes m}/\sim$ where $a\otimes b \sim -(-1)^{|a||b|}\, b\otimes a$

We will write

$$\mathrm{Sym}(V) = \bigoplus_{m\geq 0} \mathrm{Sym}^m(V) \qquad \widehat{\mathrm{Sym}}(V) = \prod_{m\geq 0} \mathrm{Sym}^m(V)$$

(graded) polynomial ring generated by V

(graded) formal power series ring generated by V

<u>Prop</u>: We have $\Lambda^k(V[1]) \simeq \mathrm{Sym}^k(V)[k]$

this is a very helpful exercise.

- CE complex for DGLA

Let $(\mathfrak{g}, d, [-,-])$ be a DGLA. Let

$$C^\bullet(\mathfrak{g}) := \mathrm{Sym}^\bullet(\mathfrak{g}^\vee[-1])$$

Since $\mathfrak{g}^\vee[-1] = (\mathfrak{g}[1])^\vee$, we can think about $C^\bullet(\mathfrak{g})$ as (polynomial) functions on $\mathfrak{g}[1]$.

When $\mathfrak{g}$ is a Lie algebra,

$$C^k(\mathfrak{g}) = \mathrm{Sym}^k(\mathfrak{g}^\vee[-1]) \simeq \Lambda^k \mathfrak{g}^\vee[-k]$$

this is $\Lambda^k \mathfrak{g}^\vee$ sitting at degree k.

We get the usual CE.

Let $d_\mathfrak{g} : \mathfrak{g}^\vee[-1] \longmapsto \mathfrak{g}^\vee[-1]$ be the dual of

$$d : \mathfrak{g} \longmapsto \mathfrak{g}$$

Let $d_{[-,-]} : \mathfrak{g}^\vee[-1] \mapsto \mathrm{Sym}^2(\mathfrak{g}^\vee[-1]) \simeq \Lambda^2 \mathfrak{g}^\vee[-2]$

be the dual of the bracket

$$[-,-] : \Lambda^2 \mathfrak{g} \mapsto \mathfrak{g}$$

Note that both $d_\mathfrak{g}$ and $d_{[-,-)}$ have $\deg = 1$ (check!)

Since $C^\cdot(\mathfrak{g})$ is freely generated by $\mathfrak{g}^\vee[-1]$,

we can extend $d_\mathfrak{g}$ and $d_{[-,-)}$ to $C^\cdot(\mathfrak{g})$ by

{ • on the generator $\mathfrak{g}^\vee[-1]$, defined above

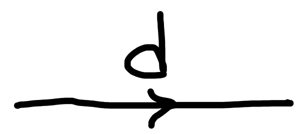
• satisfy graded Leibnitz rule.

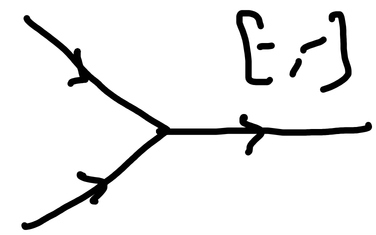
Define the CE differential

$$d_{CE} = d_\mathfrak{g} + d_{[-,-]}$$

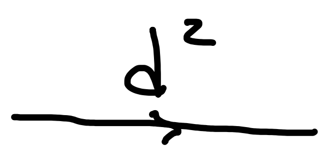
Claim: $\boxed{d_{CE}^2 = 0}$

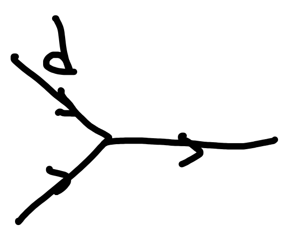
We illustrate why this is true and leave the details to readers. In fact, if we represent

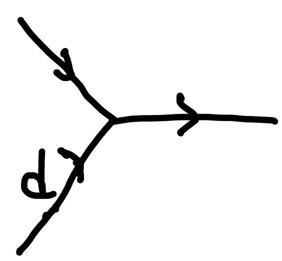
d_{CE} : 

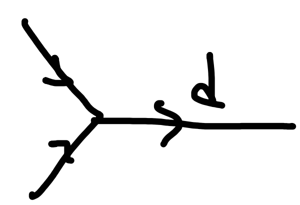
+ 

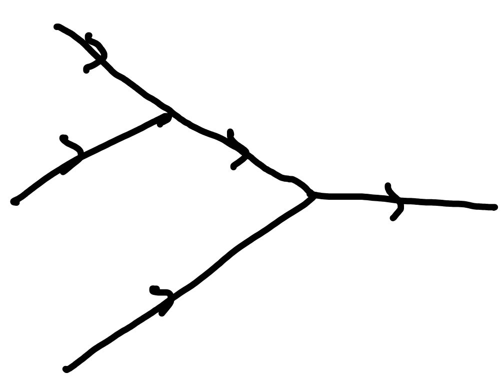
then

d_{CE}^2 : 

+ 

+ 

+ 

+ 

then we can "see" that

$$d_{CE}^2 = 0 \iff \text{defining properties of DGLA}$$

$(C^\bullet(\mathfrak{g}), d_{CE})$ is called the CE complex.

· Homotopy Lie algebra (L_∞-algebra)

Given a graded vector space V, we consider a (graded) derivation on $\mathrm{Sym}(V)$

$$\delta : \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(V)$$

which satisfies the graded Leibnitz rule

$$\delta(a \otimes b) = (\delta a) \otimes b \pm a \otimes \delta b$$

Such δ is completely determined by how δ acts on the generator

$$\delta : V \longmapsto \mathrm{Sym}(V)$$

We can decompose

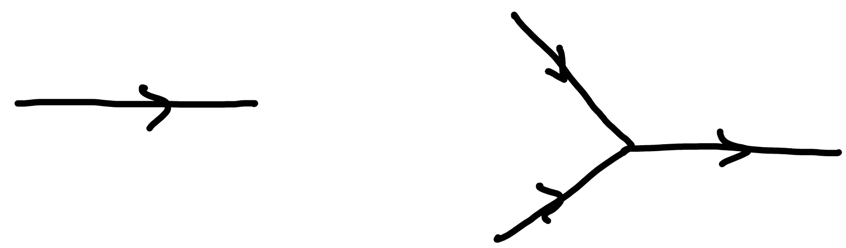
$$\delta = \delta_0 + \delta_1 + \delta_2 \cdots$$

where $\delta_k : V \longmapsto \mathrm{Sym}^k(V)$

For DGLA, we have

$$d_{CE} \circlearrowright C^{\cdot}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}^{\vee}[-1])$$

$$\begin{array}{c} \| \\ \underset{\delta_1}{d_{\mathfrak{g}}} + \underset{\delta_2}{d_{[-,-]}} \end{array}$$



this is a derivation where only δ_1, δ_2 are nontrivial.

It is natural to generalize by encoding all possible components δ_k. This is L_∞-algebra.

<u>Def'n</u>. An L_∞-algebra is a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ w/. a collection of multi-linear maps

$$(n \geq 1) \quad \ell_n : \Lambda^n \mathfrak{g} \longmapsto \mathfrak{g} \qquad \deg(\ell_n) = 2-n$$

Satisfying the following L_∞-relations

$$\sum_{k=1}^{n} \pm \ell_{n-k+1}\left(\ell_k(-,-,--),-,-,\cdots-\right)=0 \quad (\forall n)$$

The complicated L_∞-relation can be understood as follows. Let

$$\delta_n : \mathfrak{g}^\vee[-1] \longmapsto \mathrm{Sym}^n(\mathfrak{g}^\vee[-1]) \simeq \Lambda^n(\mathfrak{g}^\vee)[-n]$$

denote the dual of ℓ_n. Note that

$$\deg(\ell_n)=2-n \iff \deg(\delta_n)=\underline{1}$$

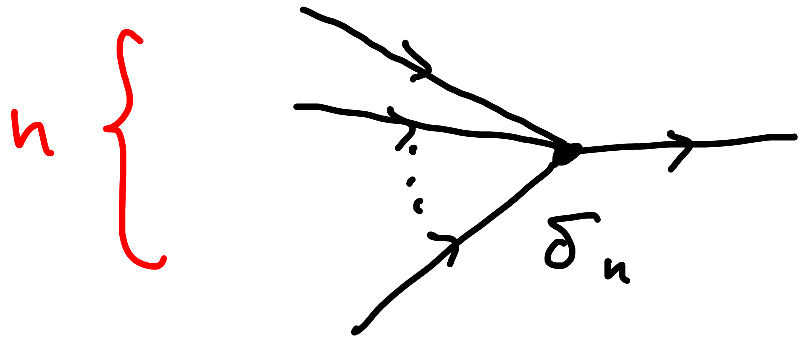
Let $\delta = \sum_{n\geq 1} \delta_n = \delta_1+\delta_2+\cdots$

then δ defines a derivation on $\underline{C^\cdot(\mathfrak{g})=\widehat{\mathrm{Sym}}(\mathfrak{g}^\vee[-1])}$ via the graded Leibnitz rule. Then

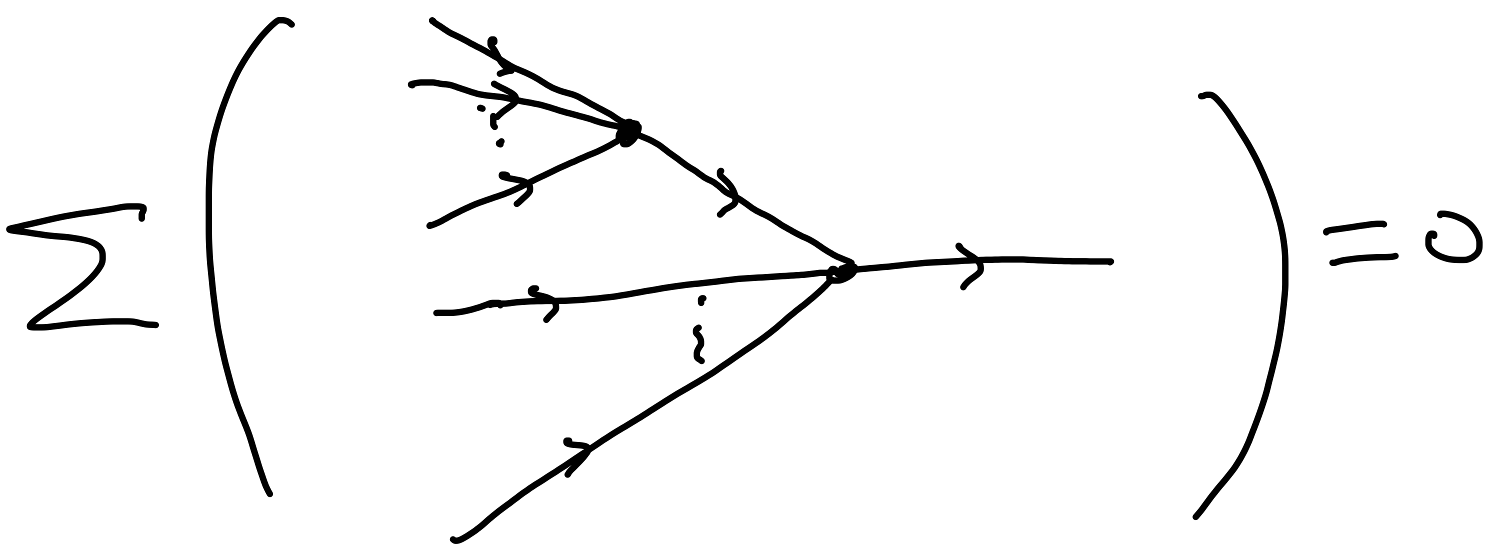
Here we use formal power series so δ is defined.

$$L_\infty\text{-relations for } \{\ell_n\}_{n\geq 1} \iff \boxed{\delta^2=0}$$

If we represent each δ_n as a graph



Then $\delta^2 = 0$ can be pictured as



As we will see, this has a natural interpretation via Feynman Diagram technique.

<u>Ref Today</u>:

- Lada, Stasheff : The Resurgence of L_∞-structures in field theory

Nice review on L_∞-alg and its relation in QFT (references there)

- Li, Zeng : Homotopy algebras in higher spin theory

Contains self-contained review on L_∞ and we follow that